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Bachelor Thesis in Physics
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On Cosmic Inflation and Large-Scale Structure's Quantum Seeds

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Abstract (de): Diese Arbeit untersucht kosmische Inflation als mögliche Quelle der primordialen Dichteschwankungen, die bekanntlich zur Entstehung von großräumigen Strukturen im Universum geführt haben. Die Entwicklung dieser Dichtestörungen ist weitgehend bekannt, jedoch ist ihr Ursprung bis heute ungeklärt. Guths Idee der Inflation im frühen Universum [1] bietet eine mögliche Lösung dieses Problems. Hierbei würden quantenmechanisch bedingte Fluktuationen auf makroskopische Skalen gestreckt; die Saatkerne für die Strukturentwicklung wären somit das Ergebnis fundamentaler physikalischer Prozesse. Inflation hat sich in den letzten beiden Jahrzehnten als universelle Lösung vieler kosmologischer Probleme etabliert. Eine inflationäre Phase im frühen Universum mit hoher Wahrscheinlichkeit nachzuweisen wäre ein großartiger Erfolg für das kosmologische Standardmodell. Deshalb werden außerdem Spuren, die ein inflationäres Universum hinterlassen haben könnte, untersucht. Diese kann man unter Umständen heute, in einer Zeit der „Präzisionskosmologie“, nachweisen.

Abstract (en): This thesis is dealing with inflation as a feasible source of primordial density perturbations, which are believed to have spawned the universe's large scale structure as we know it today. The evolution of these perturbations is generally well understood, unlike their origin. Guth's idea of an inflationary phase of the universe [1] provides a possible answer. It would imply that fluctuations due to quantum uncertainty were stretched to macroscopic scales. Those seeds for further structure evolution would therefore be the direct result of fundamental physical processes. Inflation has established itself as universally applicable solution to a myriad of cosmological problems. An observational verification of an inflationary phase in the early universe would be a massive success for the cosmological standard model. Hence, I shall examine traces, which an inflationary universe might leave for us to observe, in today's "precision cosmology" era.

„Zwei Dinge erfüllen das Gemüth mit immer neuer und zunehmender Bewunderung und Ehrfurcht, je öfter und anhaltender sich das Nachdenken damit beschäftigt: der bestirnte Himmel über mir und das moralische Gesetz in mir. Beide darf ich nicht als in Dunkelheiten verhüllt, oder im Überschwenglichen, außer meinem Gesichtskreise suchen und blos vermuthen; ich sehe sie vor mir und verknüpfe sie unmittelbar mit dem Bewusstsein meiner Existenz.“

–Immanuel Kant, *Kritik der Praktischen Vernunft*

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1. Introduction

1.1. Introductory Remarks

Few other fields in physics have seen such remarkable changes in the last 20 or so years than cosmology. In the last 10 years, the term "precision cosmology" [2] has arisen. And this is certainly not far-fetched, since, for the first time in history, our knowledge of the universe as a whole has caught up with other fields in science, reducing the error bars on fundamental parameters down to $\leq \mathcal{O}(10^{-2})$. Cosmology, in its current form, is a rather young branch of physics. It is based upon the mathematical framework of general relativity, so it emerged in the first half of the 20^{th} century. Like other fields of physics, it lives and breathes the duality of theory and experiment. Here however, the "experiments", like in other branches of astrophysics, are clearly limited to observations and simulations.
Nevertheless, the leaps in technology in the last two decades made it possible not only to conduct high precision observation campaigns, but also to carry out numerical simulations of unseen magnitude and accuracy. What used to be done by a data processing centre can now be achieved by an inexpensive consumer machine.
In the last decades, a now broadly accepted model for our universe has been established. It is based upon simple and sensible assumptions, which I will present in this section. I shall furthermore give a fairly brief overview of this so-called Cosmological Standard Model.

1.2. The Cosmological Standard Model

General relativity plays a fundamental role in cosmology; it can indeed be stated that, without general relativity in its current form, it would be impossible to have quantitative physical cosmology. A space-time model in GR is mostly defined via its metric tensor $g_{\mu\nu}$, which carries the geometry information. There are a few of those models that I'm going to refer to in this thesis, so let's familiarise ourselves with them.

1.2.1. Robertson-Walker Metric & The Friedmann Equations

The most important model in cosmology is the one defined by the *Friedmann-Lemaître-Robertson-Walker Metric*, which expresses the cosmological principle in the formalism of general relativity. I shall repeat it for the sake of completeness:

Cosmological Principle

The Universe is – on sufficiently large scales – *homogeneous* and *isotropic*.

In the most general case, the line element of the FLRW-Metric takes this form:

$$ds^2 = -dt^2 + a(t)^2[dr^2 + f_k(r)d\mathbf{\Omega}] \quad (1)$$

Where $f_k(r)$ is a function that depends on the curvature k of space-time. In particular:

$$f_k(r) = \begin{cases} k^{-\frac{1}{2}} \sinh\left(rk^{\frac{1}{2}}\right) & \text{for } k > 0, \\ r & \text{for } k = 0, \\ -k^{-\frac{1}{2}} \sin\left(r - k^{\frac{1}{2}}\right) & \text{for } k < 0. \end{cases} \tag{2}$$

f_k can be specialised to $k = \{-1,0,1\}$ if we transplant the curvature radius information to the radial coordinate r.

$a(t)$ is the so-called scale factor. Note $a_0 \equiv a(now) = 1$.
Requiring isotropy necessitates a not to depend on any spatial coordinates, while homogeneity means here that we can choose our coordinates' point of origin freely.
H, the so-called Hubble constant, is defined as the change in a in units of a itself: $H = \frac{\dot{a}}{a}$
Using this metric, Einstein's field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} - g_{\mu\nu}\Lambda \tag{3}$$

reduce to the *Friedmann equations* [3]:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right) + \frac{\Lambda}{3}. \tag{5}$$

Differentiating (4) w.r.t. coordinate time and plugging it into (5) gives

$$-\dot{\rho} = 3H\left(\rho + p\right). \tag{6}$$

This is obviously a continuity equation; it describes the time-evolution of a density ρ.
Equations (4), (5) and (6) are crucially important to today's cosmological standard model.
ρ is the energy density, so it usually includes matter and radiation; basically anything that interacts with spacetime. It can also be written as $\rho_{total} = \rho_M + \rho_\gamma + \rho_\Lambda$, where the latter corresponds to the cosmological constant Λ.

1.2.2. De Sitter Space

Consider (4) rewritten as

$$\dot{a}^2 - \left(\frac{8\pi G}{3}\rho + \frac{\Lambda}{3}\right) a^2 = -k. \tag{7}$$

Let's assume for the moment that $\rho = 0$, meaning that there is no matter or radiation and our model universe is dominated by a cosmological constant alone. If we now let a grow sufficiently, $\dot{a}$ and a are still comparable, while k will become negligible. We obtain $\dot{a}^2 = \frac{\Lambda}{3}a^2$, and thus

$$a \propto \exp\left(\sqrt{\frac{\Lambda}{3}}\,t\right) \tag{8}$$

This expansion rate will become important later, when inflation is discussed. There, the universe becomes nearly desitterian.

1.2.3. The Cosmic Microwave Background Radiation

A hot big bang produces a perfect black-body spectrum of radiation. The mean temperature of the photons, however, drops over time, since the space they occupy expands. At early times, these background photons are constantly scattered off free electrons due to Thomson scattering; when the electrons (re-)combined with protons to form helium, however, the universe suddenly became transparent for this radiation. In the course of $\sim 30{,}000$ years, the cosmos went from completely opaque to practically no absorption. Today, the temperature of this radiation can be precisely measured: $T_{cmb} \simeq 2.7K$. The all-sky map of this radiation has the same temperature everywhere, give or take deviations of $\mathcal{O}(10^{-4})$. Analysing the cosmic microwave background is one of the most important assets in an observational cosmologist's toolset.

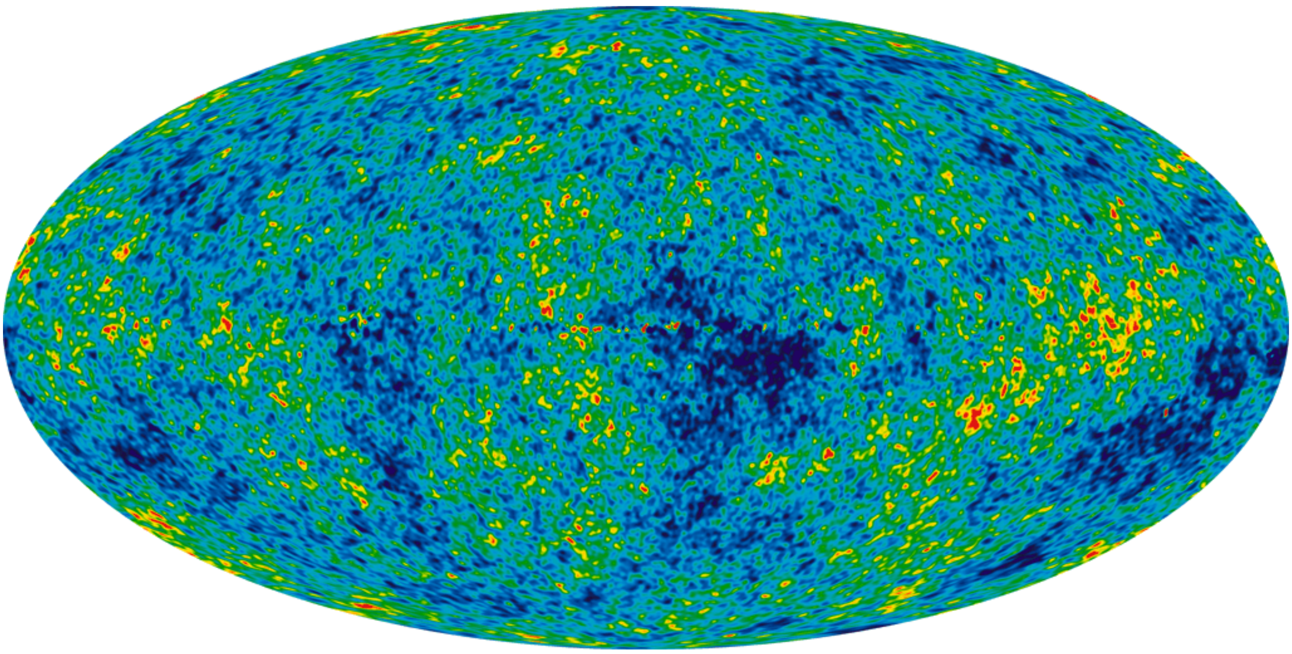

Figure 1: All-sky map of the CMBR, taken by the WMAP satellite over the course of seven years. The colour map corresponds to fluctuations of $\pm 200\mu K$ around the mean Temperature of $T = 2.725K$. The cosmic microwave background is an extremely rich source of information for cosmologists; ground-based, air-borne and space-borne measurements have dramatically decreased the error bars of cosmological parameters over the last 20 years. With the Planck satellite currently doing its second full-sky sweep, more precise data can be expected within the next few years. [4],[5]

1.2.4. Density Parameters

By setting $k = 0$ in (4), and solving it for $\rho_{total} = \rho_M + \rho_\gamma + \frac{\Lambda}{8\pi G}$, one derives the so-called critical density

$$\rho_{crit} = \frac{3H^2}{8\pi G} \tag{9}$$

The density parameters are now defined as the relative contribution of each component of ρ_{crit} :

$$\Omega_M := \frac{\rho_M}{\rho_{crit}} \quad \Omega_\gamma := \frac{\rho_\gamma}{\rho_{crit}} \quad \Omega_\Lambda := \frac{\frac{\Lambda}{8\pi G}}{\rho_{crit}} = \frac{\Lambda}{3H^2}$$

Their evolution over time can be derived from (6). Using the equation-of-state parameter $w = \frac{p}{c^2\rho}$, it reads

$$-\frac{\dot{\rho}}{\rho} = 3\left(\frac{\dot{a}}{a}\right)(1+w) \tag{10}$$

and therefore implying

$$\rho = \rho_0 a^{-3(1+w)}. \tag{11}$$

For relativistic particles we have $w = \frac{1}{3}$, for "cold" matter $w = 0$ and for the cosmological constant $w = -1$.
This leads to

$$\begin{aligned} \rho_\gamma &= \rho_{\gamma,0} \cdot a^{-4} \\ \rho_M &= \rho_{M,0} \cdot a^{-3} \\ \rho_\Lambda &= \tfrac{\Lambda}{8\pi G} \propto a^0 \end{aligned}$$

See Fig.2 for a quick-and-dirty numerical approximation of their evolution. So that now (4) can be easily rearranged to read

$$H^2 = H_0^2\left(\Omega_{\gamma,0} \cdot a^{-4} + \Omega_{M,0} \cdot a^{-3} + \Omega_{\Lambda,0} + \Omega_K \cdot a^{-2}\right), \tag{12}$$

where $-\frac{Kc^2}{H_0^2} = \Omega_K$.
Combining the 7-year WMAP cosmic microwave background data and a couple of other sources (2dFS, SDSS, HST KP, etc.) and fitting them to a ΛCDM-Model with $\Omega_K = 0$ yields the following[6]:

Parameter	Value	Error (1σ)
h	0.704	± 0.14
$\Omega_{b,0}$	0.0456	± 0.0016
$\Omega_{cdm,0}$	0.227	± 0.014
$\Omega_{M,0}$	0.273	± 0.0141
$\Omega_{\Lambda,0}$	0.728	± 0.016

These values also agree with other ways of approximating them, e.g. via the abundance of helium due to the big-bang nucleosynthesis, weak lensing measurements, etc. The fact that using conceptually completely different ways of measuring those parameters, and coming up with similar results, is nothing short of amazing; it confirms that we are on the right track and that our theories are – at least – not completely off.

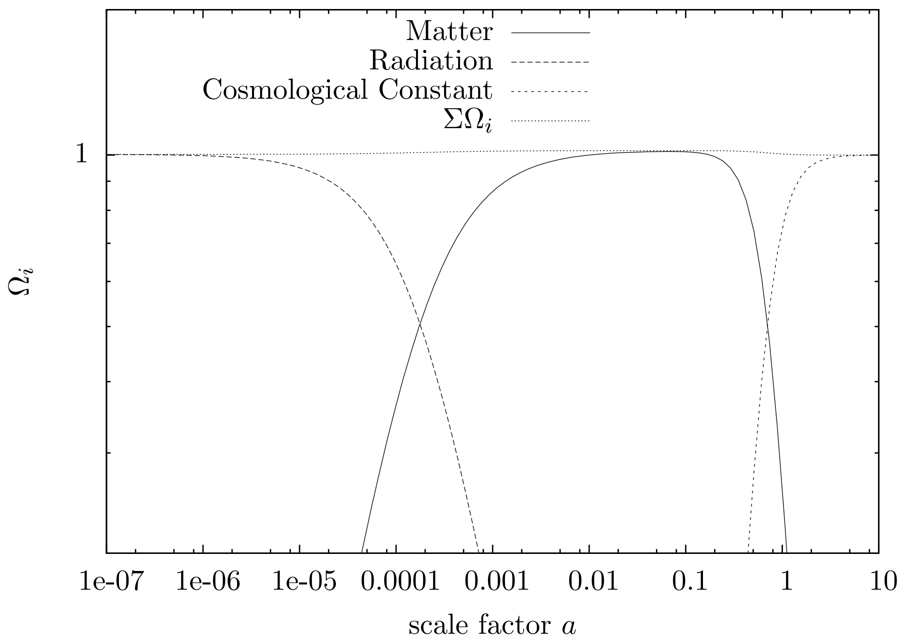

Figure 2: The evolution of Ω_i with scale factor a. Obviously, there are regions where different components dominate the universe, therefore they are called radiation-dominated and matter-dominated respectively. At $a_{eq} \approx 1.8 \times 10^{-4}$, Ω_M and Ω_γ are equal; this point called matter-radiation equality. We are also currently entering an era of "Λ-domination", where expansion is accelerating. [7]

1.3. Miscellaneous definitions

1.3.1. Comoving coordinates, conformal time

Comoving coordinates are defined such that distances do not change over time, viz. distances l scale with $a(t)$. Actually, the FLRW metric already describes comoving coordinates. Note that the components are stretched or contracted by $a(t)$. This is often implied in cosmological literature when people speak of "distances". It is extremely useful in calculations and also makes things more intuitive at times.

In this context, let's also define *conformal time* as:

$$\tau(t) = \int_0^t \frac{dt'}{a(t')} \tag{13}$$

The *particle horizon* is defined as $c \cdot \tau(t)$. It is easily confused with the event horizon; the difference is that something outside our particle horizon was, so far, not able to causally affect us; something outside our event horizon, however, will *never* be able to causally affect us. In order to link $a(t)$ to some observable quantity, consider a light ray emitted at $a(t_{em})$ by a resting (i.e. comoving) source and detected at $a(t_{obs}) = a(t_0)$ by a comoving observer. The distance between both points,

$$r = \int_{t_{em}}^{t^{obs}} \frac{cdt}{a(t)} \tag{14}$$

must stay the same, so changing the integral limits must not affect the integral. This means $\frac{dt_{emit}}{dt_{obs}} = \frac{a(t_{emit})}{a(t_{obs})}$. Interpreting dt as frequency ν, and using $\nu\lambda = c$ leads to

$$\frac{\lambda_{obs}}{\lambda_{emit}} := (1+z) = \frac{a(t_{obs})}{a(t_{emit})} = \frac{1}{a(t_{emit})} \tag{15}$$

Where z is the redshift; a very much observable value. Note that this redshift is due to the expansion of the universe; the photons lose energy because their physical wavelength is getting stretched out on their journey to us.

The Hubble radius is the distance at which objects recede from us with the speed of light. It is found via

$$r_H = \frac{c}{H(z)} \tag{16}$$

1.3.2. Time dependence of $a(t)$

In order to find out how $a(t)$ scales with time, let's consider (12) with $\Omega_K = 0$:

$$H = \frac{\frac{da}{dt}}{a} = H_0 \left(\Omega_{\gamma,0} \cdot a^{-4} + \Omega_{M,0} \cdot a^{-3} + \Omega_{\Lambda,0}\right)^{\frac{1}{2}} \tag{17}$$

Separating the variables and integrating returns

$$H_0 t = \int_0^a \frac{da'}{a' \left(\Omega_{\gamma,0} \cdot a'^{-4} + \Omega_{M,0} \cdot a'^{-3} + \Omega_{\Lambda,0}\right)^{\frac{1}{2}}}. \tag{18}$$

Let's consider the limiting cases:

- During radiation domination, $\Omega_{\gamma,0} \cdot a^{-4}$ dominates the denominator, resulting in $a(t) \propto \sqrt{t}$.
- When matter dominates, $\Omega_{M,0} \cdot a^{-3}$ is the relevant term and $a(t) \propto t^{2/3}$.
- The cosmological constant takes over at late times and $a(t) \propto \exp\left(\sqrt{\frac{\Lambda c^2}{3}}\ t\right)$, as discussed before.

1.4. Conclusion

Again, this section can not at all claim to be a complete introduction into cosmology. It is merely meant to remind the reader of the main points in isotropic and homogeneous cosmology, its implications and properties, since most of what was introduced here will be used later. Much has been left out – purposely so – as not to complicate matters unnecessarily.

2. Inflation

The paradigm of inflation seems quite radical when one is confronted with it. In essence, it requires an accelerated expansion of space, during which the limits of causality are being shifted dramatically.

2.1. Shortcomings of the Standard Model

While the huge success of cosmology's standard model is quite remarkable, many questions it allows us to ask remain unanswered, at least of now. In the 1970's, the emerging standard model of cosmology was faced with a bizarre problem: The density parameters Ω_i, especially Ω_k, had to be meticulously *fine-tuned* in order to make the standard model return the universe in its current form at $t = now$. In particular the universe was required to survive long enough to be able to have developed life whilst not "freezing out" and becoming, essentially, empty. Also, the universe's astonishing homogeneity on large scale and its isotropy are by no means evident from theory. Most simple solutions to Einstein's field equations are quite far from homogeneous and not necessarily isotropic.

2.1.1. The Horizon & Flatness Problems

Among the problems mentioned above are, most famously, the horizon and flatness problems.

The observable large-scale homogeneity of the sky (see Fig. 1) is astonishing. But it is even more puzzling if one calculates the particle horizon for any given particle:

$$r_h = c \int_0^{t_0} dt/a(t) \tag{19}$$

For negligible curvature, this corresponds to an angle of about 1^o on the sky at $z = z_{rec} \sim 1000$. Regions greater than r_h could *not* exchange information since the big bang. So, the CMBR is made up of numerous regions without causal contact; why, then, is the CMBR so smooth and homogeneous on all scales? This discrepancy is called the Horizon Problem.

Consider the Friedmann equation (4), re-written as

$$\frac{3H^2}{8\pi G}a^2 - \rho_{tot}a^2 = -\frac{3kc^2}{8\pi G} \tag{20}$$

where $\rho_{tot} = \rho_\gamma + \rho_M + \rho_\Lambda$.
The rhs is just a constant, let's call it C; the lhs can be rewritten as

$$\left(\frac{\rho_{total}}{\rho_{crit}} - 1\right)\rho_{tot}a^2 = \left(\frac{1}{\Omega(a)} - 1\right)\rho_{tot}a^2 = C \tag{21}$$

Now, ρ_{tot} goes as a^{-4} during radiation domination and like a^{-3} during matter domination. This means $\rho_{tot}a^2$ is still decreasing.
In order to validate the equation, the bracket term has to increase by the same amount. For $\Omega = 1$, this is obviously not a problem; the statement then would be $C = 0$. But for $\Omega \neq 1$ – even by a tiny amount – Ω will deviate more and more from unity, increasing the curvature contribution. Obviously, we observe $\Omega_0 \simeq 1$; this requires Ω to be $1 \pm \mathcal{O}(10^{-60})$ around the Planck epoch. Such accurate

bounds on initial conditions are quite unsatisfying. Of course, one can state that Ω just happened to be in a certain interval at a specific time. However, a process that *pushes* Ω towards unity to begin with would be easier to grasp than relying on the rather tautological anthropic principle: We're here because we're here.

2.2. Constructing Inflation

2.2.1. Accelerated Expansion

Let's see what we need to construct in order for the flatness problem to disappear: If ρa^2 was to increase rather than drop, we can drive the universe towards flatness. Consider again (4):

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{22}$$

We want $\frac{d(\rho a^2)}{dt} > 0$, meaning

$$\frac{d}{dt}\rho a^2 = \frac{d}{dt}\left[\frac{3}{8\pi G}\left(\dot{a}^2 + kc^2\right)\right] > 0 \tag{23}$$

$$\Leftrightarrow \frac{3}{8\pi G}\frac{d}{dt}\dot{a}^2 > 0 \tag{24}$$

$$\Leftrightarrow 2\dot{a}\ddot{a} > 0 \tag{25}$$

$$\Leftrightarrow \ddot{a} > 0. \tag{26}$$

Here, I used the fact that the overall curvature does not depend on time; if this process takes place, the curvature radius is merely increased. Also, $\dot{a}$ is certainly always greater than 0, since we want an expanding universe.

Thus, if Ω is to be forced towards unity,accelerated expansion is required.
Let's see how $\ddot{a}$ can be greater than 0: Equation (5), with Λ contained in ρ and p, states:

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a \tag{27}$$

Therefore, $\ddot{a} > 0$ requires

$$\rho + \frac{3p}{c^2} < 0. \tag{28}$$

Since ρ is obviously positive, it is most common to require negative pressure.
Equation (27) implies $w = \frac{p}{c^2\rho} < -\frac{1}{3}$. This requirement is met, for example, by the cosmological constant ($w = -1$). As we saw in the Introduction (cf. equation (8)), a universe dominated by cosmological constant will go into exponential expansion, regardless of its curvature. (It will start out differently, but converge towards an exponential function whatsoever). Negative pressure is admittedly something counter-intuitive; but mathematically, a *scalar field* can be the solution.

Therefore, inflation resolves the flatness problem by construction, but it also solves the horizon problem. To show the latter, consider the comoving Hubble radius

$$r_{H,com} = \frac{c}{\mathcal{H}} = \frac{c}{Ha}. \tag{29}$$

Since H fairly constant during inflation (which will be shown later) and a grows exponentially, it becomes immediately obvious that $r_{H,com}$ is shrinking. This is quite peculiar; it means that the regions in causal contact are shrinking; again, something that doesn't agree well with our intuition.
But this also solves the horizon problem! While the CMBR as we see it today consists of many causally disconnected regions, they might just have been in causal contact before or during inflation, "un-causalised" by the shrinking comoving Hubble sphere.
In order to drive Ω from $\mathcal{O}(1)$ to the required limit at early times, inflation is supposed to have caused at least a factor of $e^{60} \approx 10^{26}$ of expansion [8].

2.2.2. Solution: A Scalar Field

Since we always assume ρc^2, the energy density, to be positive, we need negative pressure to meet the requirements of eq. (28). The answer to this problem is a scalar field ϕ, that can, under certain conditions, exert negative pressure.
Probably the simplest Lagrangian density that still has nontrivial dynamics one can choose[1] is

$$\mathcal{L} = \frac{1}{2}\phi_{;\mu}\phi^{;\mu} + V(\phi) \tag{30}$$

Obviously, the covariant derivatives can be replaced by partial ones, since ϕ is a scalar. ϕ is sometimes called the Inflaton field.
If this field is coupled with GR, the action becomes

$$S = \int d^4x\sqrt{g}\mathcal{L}(\phi) + S_H, \tag{31}$$

where S_H is the Hilbert action.
The energy-momentum tensor then becomes [9]:

$$T^{\alpha}_{\nu}(\phi) = g^{\alpha\lambda}\phi_{,\lambda}\phi_{,\nu} - g^{\alpha}_{\nu}\left(\frac{g^{\sigma\lambda}\phi_{,\sigma}\phi_{,\lambda}}{2} - V(\phi)\right). \tag{32}$$

Now, we can read off the energy density $\epsilon = \rho c^2 = T_{\mu\nu}u^{\mu}u^{\nu}$ and the pressure $p = \frac{T^i_i}{3}$[2]:

$$\rho c^2 = T_{\mu\nu}u^{\mu}u^{\nu} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{33}$$

$$p = \frac{1}{3}T^i_i = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{34}$$

[1] This is sometimes a feasible argument in theoretical physics. For example, the Einstein field equations can be derived by assuming the simplest action, i.e. $S_H = \int R\sqrt{-g}d^4x$ and then putting $\delta S = 0$. This is how David Hilbert found Einstein's equations before Einstein.

[2] Latin indices imply summation over the three spatial coordinates, as usual.

Here, the $g_{\mu\nu}$ was assumed to be the FLRW-metric.
The 4-velocity u^μ is normalised to $\langle u, u\rangle = g_{\mu\nu}u^\mu u^\nu = -1$ and the scalar field only depends on time ($\phi = \phi(t)$) for spatial isotropy.
Therefore, the scalar field ϕ has the equation of state parameter

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{35}$$

$w < -\frac{1}{3}$ can now be achieved if the potential term $V(\phi)$ dominates over the "kinetic" term $\frac{1}{2}\dot{\phi}^2$.

We now require $V \gg \dot{\phi}^2$ for a sufficient time to have inflation, but we also need some sort of mechanism to stop inflation in order to let the universe develop "normally" afterwards.
The potential is usually left arbitrary, left to be constrained by observations.
The *Klein-Gordon equation* is the equation of motion for a free quantum scalar field [10]:

$$(\Box + \mu^2)\phi(\vec{x},t) = 0 \tag{36}$$

$\Box$ is the d'Alembert operator: $\Box = (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)$, μ a reduced mass, namely $\mu = \frac{mc}{\hbar}$.
If $m = 0$, it obviously reduces to the wave equation.
From both the Klein-Gordon equation and the conservation equation (6) one finds:

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{37}$$

The prime denotes a derivative w.r.t. ϕ. Note the damping-term $3H$ and keep this in mind for later.
Now, plugging (35) into the curvature-free Friedmann equations yields

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{38}$$

We now have everything at hand to study a scalar field driven inflation.

2.3. Making Inflation work: Slow-Roll Approximation

During inflation, we know that $V \gg \dot{\phi}^2$ by construction, so that equations (37) & (38) from above become

$$3H\dot{\phi} + V' = 0 \tag{39}$$

and

$$H = \sqrt{\frac{8\pi G}{3}V(\phi)} \tag{40}$$

Now we can define the dimensionless slow-roll parameters

$$\epsilon := \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \tag{41}$$

$$\eta := \frac{1}{8\pi G}\left(\frac{V''}{V}\right) \tag{42}$$

And require $\epsilon, \eta \ll 1$.
This is quite intuitive; first, we want V to be roughly constant (and thus H as well), so we want it to change slowly with ϕ, which is a function of time alone.
Then, because inflation has to last long enough, we require that the "acceleration" of V to be small.
Inflation ends as soon as one of the slow-roll parameters approaches unity.

2.4. Visualising Inflation

Often, visualising something helps us humans getting our head around a complicated matter. And when dealing with something as complicated as the universe, this is clearly the case.

2.4.1. The Hubble Sphere

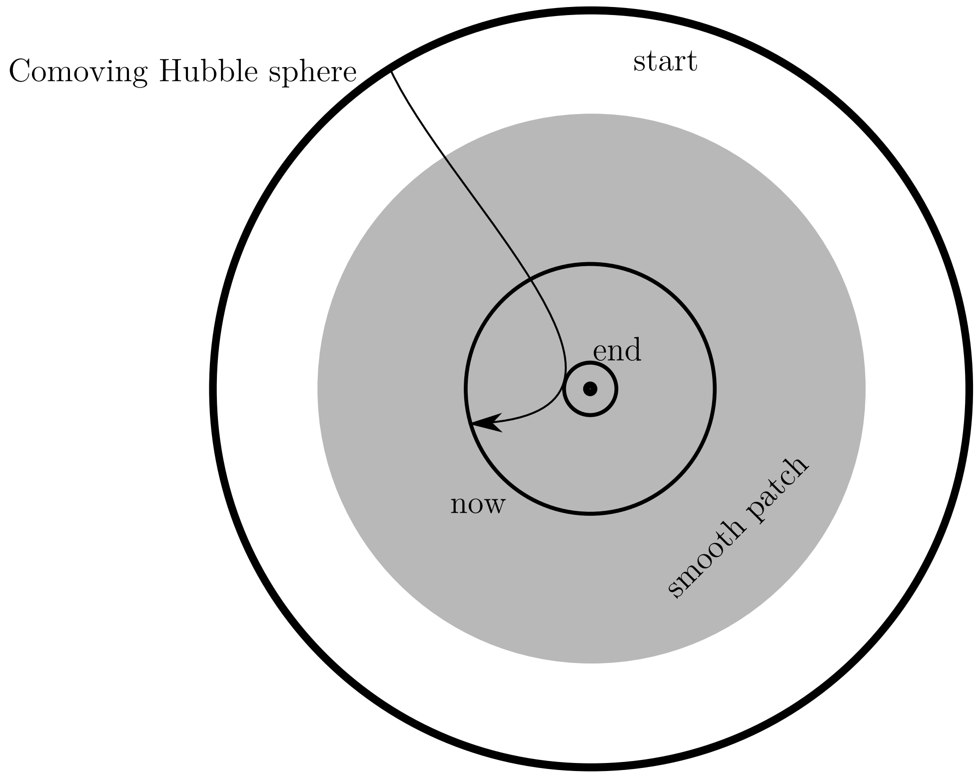

Figure 3: This figure, reproduced from Liddle & Lyth [11, Figure 3.2], shows the comoving Hubble radius at the start of inflation, where it is much bigger than a region called "smooth patch", parts of which humans will later observe to be homogeneous. It shrinks to a minimal size, and, at the end of inflation, starts to grow again until it reaches its current size. The Observer in the centre perceives the observable universe (i.e. the Hubble sphere) to be miraculously homogeneous.

2.4.2. Inflation in Minkowski Diagrams

Minkowski diagrams are arguably the best form of visualisation for things like particle horizons, event horizons, etc. Whenever one is interested in causality, they make things easy: Time-like and space-like distances are clearly distinguishable.

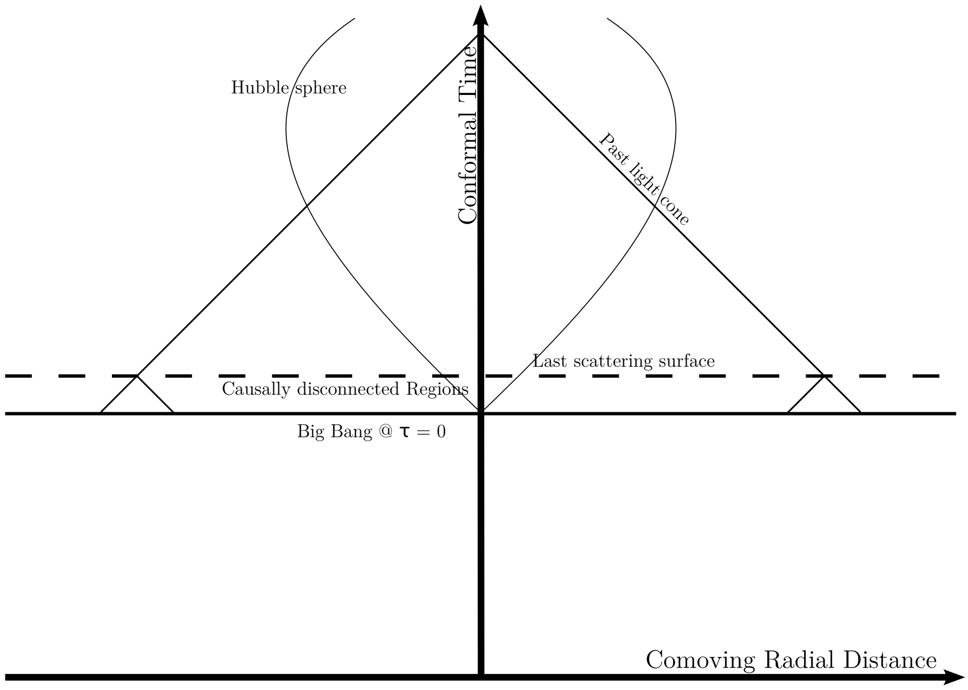

Figure 4: Here, we see the homogeneity problem in a nutshell. The observer sees a nice and homogeneos CMB, but the scattering surface's past light cone is far, far too small to explain the large-angle homogeneity we perceive. Note the Hubble sphere, that is shrinking at later conformal times due to Ω_Λ taking over. This plot is a simplified version reproduced from Longair [12].

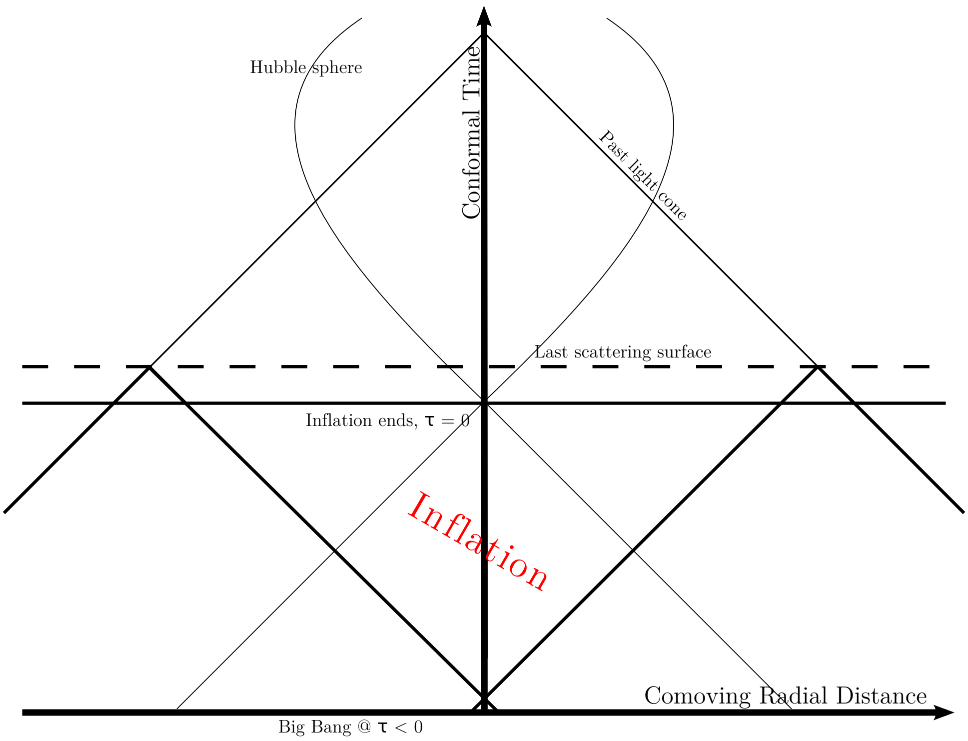

Figure 5: This works much better. If we set conformal time τ to 0 at the end of inflation, and extrapolate the light cones, the last scattering surface will be in causal contact (i.e. the light cones intersect). The Hubble sphere is shrinking during inflation and starts to expand again at $\tau = 0$.

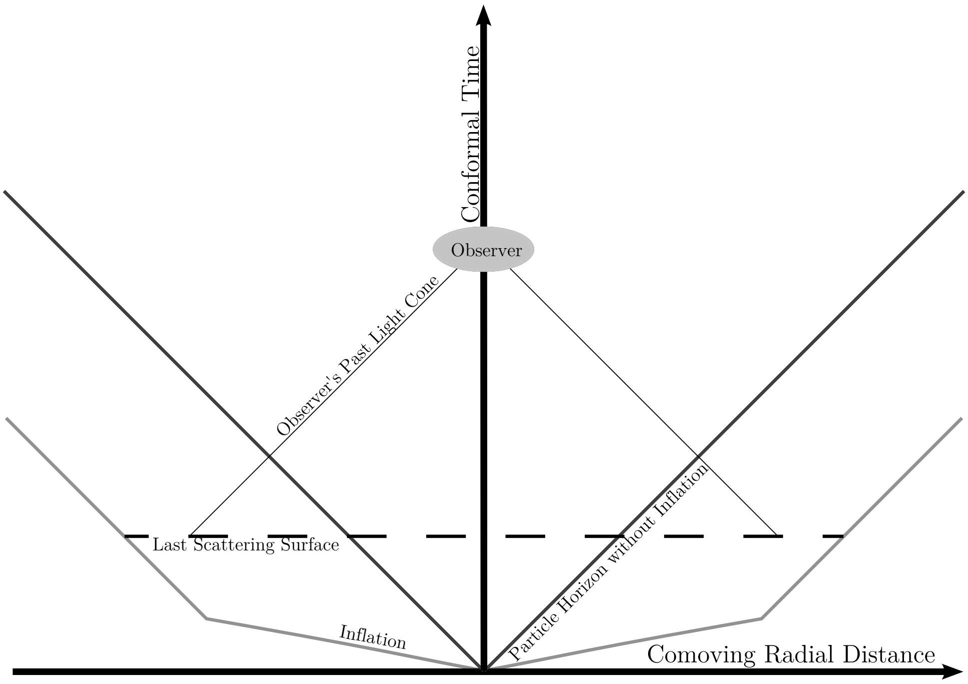

Figure 6: A visualisation of inflation reproduced from [9]. One can see clearly how cosmic inflation solves the horizon problem. The last-scattering surface is in casual contact if inflation occurred, while the standard particle horizon does not account for the CMB's homogeneity. Note that this sketch is not to scale; a slope of 45^o, however, indicates light-like worldlines. Inflation, therefore, has a space-like worldline; this is not to be confused with superluminal motion: We're looking at the co-moving distance here; for more on the subject, see [13] for a very helpful description of common cosmology pitfalls.

3. Structure's Quantum Seeds

In quantum field theory, vacuum is far from empty. There is constant pair production and annihilation going on; the energy necessary for production is "borrowed" from the vacuum and then given back at annihilation. This process is due to the Heisenberg uncertainty principle and those particles are merely virtual.

This can be used to qualitatively explain Hawking radiation of a black hole: Near the event horizon, a particle-antiparticle pair is produced. One of them gets pulled into the black hole and is beyond the event horizon; the other therefore becomes a real particle and, if it has enough momentum to reach an observer, one measures radiation coming from the direction of the black hole. The same can be applied to de-Sitter-spacetime with constant horizon $\frac{1}{H}$: A particle-antiparticle pair is produced within the horizon, but the expansion, obviously accelerating, drives them away from each other quickly, making it impossible for them to annihilate.

This is the basic idea behind inflation as a source of the initial density perturbations, which went on to form clusters, galaxies and stars; and of course a particularly curious form of life.

3.1. The Quantum-Mechanical Harmonic Oscillator

Before actually dealing with the Inflaton's perturbations, let's review a well-known quantum mechanics problem: The 1-dimensional harmonic Oscillator. The equation of motion is given by

$$\ddot{x} + \omega^2 x = 0. \tag{43}$$

Where ω can explicitly depend on time; this will become important later. Its Hamiltonian is

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2, \tag{44}$$

and the commutator

$$[\hat{x},\hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \tag{45}$$

If we now go into the Heisenberg picture, and assume unit mass for simplicity ($p = m\dot{x} \rightarrow p = \dot{x}$), we can write

$$[\hat{x},\hat{p}] = [x,\dot{x}] = i\hbar \tag{46}$$

To obtain the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, let's decompose $\hat{x}$:

$$\hat{x}(t) = u(t)\hat{a} + u^*(t)\hat{a}^\dagger \tag{47}$$

Obviously, $u(t)$ has to satisfy equation (43). It is called the mode function.

For the commutator we now have

$$[x(t), \dot{x}(t)] = (u\dot{u}^* - \dot{u}u^*)[\hat{a},\hat{a}^\dagger] = i\hbar, \tag{48}$$

and defining $\langle f, g\rangle \equiv \frac{1}{i\hbar}(g\dot{f}^* - \dot{g}f^*)$ for arbitrary functions f and g, we get

$$\langle u, u\rangle[\hat{a},\hat{a}^\dagger] = 1. \tag{49}$$

We can now choose u such that it is normalised in the sense that $\langle u, u\rangle = 1$; this leads to the known commutator for a and $a^\dagger$:

$$[\hat{a},\hat{a}^\dagger] = 1 \tag{50}$$

Using the $\langle\cdot,\cdot\rangle$–definition from above, the ladder operators can be written thusly:

$$\hat{a} = \langle u, \hat{x}\rangle \tag{51}$$

$$\hat{a}^\dagger = -\langle u^*, \hat{x}\rangle \tag{52}$$

Let's assume for a minute that ω is not time-dependent. In fact, if $\omega = \omega(t)$, we are faced by a bit of a dilemma. Because, with ω not fixed in time, there is no unique solution for $u(t)$, and thus no unique solution for the vacuum state.
This shall be picked up later.
So with time-invariant ω, the Hamiltonian acting on the ground state $|0\rangle$ gives us

$$\hat{\mathcal{H}}|0\rangle = \frac{(\dot{u}^2 + \omega^2 u^2)^*}{2}\hat{a}^\dagger\hat{a}^\dagger + \frac{1}{2}\left(|\dot{u}|^2 + \omega^2|u|^2\right)|0\rangle. \tag{53}$$

Here, $a|0\rangle = 0$ (i.e. the definition of the vacuum state) was used as well as equation (49).
Furthermore, since $|0\rangle$ is an eigenstate of $\mathcal{H}$, we require the first term to vanish, meaning

$$\dot{u} = -i\omega u \tag{54}$$

The negative sign comes from requiring $\langle u, u\rangle$ to be positive definite. Plugging this into $\langle u, u\rangle = 1$ results in

$$u(t) = \sqrt{\frac{\hbar}{2\omega}}\exp(-i\omega t) \tag{55}$$

And the Hamiltonian is in its well-known form for a quantum harmonic oscillator:

$$\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{56}$$

with its zero-point energy $\hat{\mathcal{H}}|0\rangle = \frac{\hbar\omega}{2}|0\rangle$.

Let's now try to calculate the mean square expectation value of $\hat{x}$ in the $|0\rangle$ state.

$$\langle 0|\hat{x}^\dagger\hat{x}|0\rangle = \langle 0|(u^*\hat{a}^\dagger + u\hat{a})(u\hat{a} + u^*\hat{a}^\dagger)|0\rangle \tag{57}$$

$$\Leftrightarrow \langle 0|\hat{x}^\dagger\hat{x}|0\rangle = |u|^2\langle 0|\hat{a}\hat{a}^\dagger|0\rangle \tag{58}$$

$$\Leftrightarrow \langle 0|\hat{x}^\dagger\hat{x}|0\rangle = |u|^2\langle 0|0\rangle = |u|^2 = \frac{\hbar}{2\omega} \tag{59}$$

This is why we did the decomposition of $\hat{x}$ into u. Now we have expressed $\langle|\hat{x}|^2\rangle$ in terms of u; it will soon become clear why this is important.

3.2. Perturbing the Inflaton Field

This is where things get interesting, but also very complicated. The calculations are very lengthy and taking a considerable effort to go through. For a thorough account see [14].
I will however stick to [15], which offers all of the physics involved while not being algebraically overloaded. It ignores, for example, perturbations in the metric, and is a fairly heuristic approach.

3.2.1. Decomposing ϕ

As usual in perturbation theory, our ansatz will be

$$\phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x}) \tag{60}$$

and we're interested in the evolution of $\delta\phi(t, \vec{x})$.
The Klein-Gordon equation was

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V' = 0 \tag{61}$$

Obviously, the ∇^2 term here is not dropped due to the spatial dependence of $\delta\phi$ and we are indeed interested in this perturbed part of ϕ.
Now, going into Fourier space makes thinks much easier:

$$\delta\phi_k = \frac{1}{\sqrt{(2\pi)^3}} \int d\vec{x} \left(\delta\phi(t,\vec{x}) \cdot e^{i\vec{k}\cdot\vec{x}}\right), \tag{62}$$

because equation (61) then reads

$$\delta\ddot{\phi}_k + 3H\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k - V''\delta\phi_k = 0. \tag{63}$$

Requiring slowroll, i.e. $\eta \ll 1 \Rightarrow H^2 \propto V \gg V''$, the V'' term can be neglected.
Introducing the *Mukhanov variable* $v = a\delta\phi$ and changing the time derivatives ($\frac{d}{dt} = \dot{}$) to conformal time derivatives ($\frac{d}{d\tau} = {}'$) yields "the master equation of inflationary quantum perturbation theory" [15]:

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0 \tag{64}$$

This is the harmonic oscillator with time-dependent frequency. It can be quantised easily, so that it also applies to quantum-mechanical processes.

3.2.2. Investigating different limits

First, let's look at scales much smaller than the horizon (i.e. $k \gg aH$). In that case, the $a''/a = 2H^2a^2$ is dominated by the k^2 and can thus be left out:

$$v_k'' + k^2 v_k = 0 \tag{65}$$

The solution is a harmonic oscillator, as seen above, with

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{66}$$

The perturbations are oscillating within the horizon, at least until our assumption breaks down.

Therefore, let's visit the other extreme, namely $k \ll aH$; the equation of motion becomes

$$v_k'' - \frac{a''}{a} v_k = 0; \tag{67}$$

this has a growing solution: $v_k \propto a$
Recall the definition of v:

$$\delta\phi_k = \frac{v_k}{a} \propto a^0 = const. \tag{68}$$

So on scales larger than the horizon, $\delta\phi_k$ doesn't grow, this is often called the "freezing out" of modes and is exactly what was predicted at the beginning of the chapter.

3.2.3. Bunch-Davies Vacuum

In de Sitter spacetime, equation (64) can actually be solved analytically; rewriting the factor a''/a as $2/\tau^2$ gives

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0 \tag{69}$$

Considering the boundary conditions (i.e. what v_k has to look like in the sub-horizon limit), this is solved by the Bunch-Davies mode functions

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right). \tag{70}$$

However, it is not clear whether this is a unique solution [15].

3.2.4. Zero-Point Fluctuations in the Inflaton

Now calculating the zero-point fluctuations is straightforward; remember, we are in the superhorizon limit ($k \ll aH \Leftrightarrow k\tau \ll 1$), since the sub-horizon modes are oscillating; however, as aH shrinks, smaller and smaller Wavelengths $\lambda \propto k^{-1}$ are freezing out. Therefore, using the definition of τ and equation (59), we obtain:

$$\langle 0 || \delta\hat{\phi}_k|^2 | 0 \rangle = \frac{|v_k|^2}{a^2} = \frac{H^2}{2k^3}, \tag{71}$$

Where $|v_k|^2 = \frac{1}{2k}$ was used, while $a = k/H$ at horizon crossing. To recover the result in real space, we transform $\delta\phi_k$:

$$\langle 0 || \delta\hat{\phi}|^2 | 0 \rangle = \int d\vec{k} \cdot d\vec{k'} \langle \delta\phi_k \delta\phi_{k'} \rangle e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = \int d\vec{k} \cdot d\vec{k'} \frac{H^2}{2k^3} \delta(\vec{k} - \vec{k}') e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \tag{72}$$

$$\Leftrightarrow \langle 0||\delta\hat{\phi}|^2|0\rangle = \int d\vec{k}\frac{H^2}{2k^3} \propto H^2 \tag{73}$$

We obtained tangible results for the zero-point fluctuations of the scalar field ϕ. Now we need to translate those frozen fluctuations into to the matter distribution and we have generated density perturbations from quantum oscillations. In order to do this, a strikingly simple method is the Time-Delay Formalism.

3.3. Time-Delay Formalism

Developed very early on (published in 1982!) by Guth and Pi [16], its assumption is that ϕ itself determines the end of inflationary behaviour and therefore regions where ϕ happens to be bigger will 'inflate' longer than regions with small ϕ due to the potential term.
So, a local perturbation is translated into a time-delay of

$$\delta t = \frac{\delta\phi}{\dot{\phi}} \propto \frac{H}{\dot{\phi}}, \tag{74}$$

since $\delta\phi \propto H$ (see eq. (73)).
After inflation, the universe behaves normally: $\rho \propto H^2$ (cf. (4)), so that

$$\frac{\delta\rho}{\rho} \propto 2\frac{\delta H}{H} \tag{75}$$

Keeping in mind that $H \equiv \frac{\dot{a}}{a} \propto \frac{1}{t}$ (if $a(t) \propto t^\alpha$), we get:

$$2\frac{\delta H}{H} \propto -H\delta(H^{-1}) \propto -H\delta t. \tag{76}$$

Inserting δt from (74):

$$-H\delta t \propto -\frac{H^2}{\dot{\phi}} = -\frac{H^3}{H\dot{\phi}}, \tag{77}$$

Using $H \propto \sqrt{V}$ (cf. (40)) in the numerator and using slowroll ($H\dot{\phi} \propto -V'$, see (39)) in the denominator, with the definition of $\epsilon \propto \left(\frac{V'}{V}\right)^2$ (cf. (41)), we can link this to one of the slowroll parameters:

$$-\frac{H^3}{H\dot{\phi}} \propto \frac{V^{\frac{3}{2}}}{V'} = \sqrt{\frac{V}{\epsilon}}. \tag{78}$$

So, in the end, we are left with

$$\left(\frac{\delta\rho}{\rho}\right)^2 \propto \frac{V}{\epsilon}. \tag{79}$$

This result is quite important. It means that by observing the density fluctuations, we can make conclusions about V and ϵ. If we can measure one of them by some other means, the other one is immediately determined. Usually, this equation is evaluated when modes leave the horizon (again, see [15]). Peacock [8] also gives a well-structured overview of this.

4. Observing Inflation

Since in cosmology observing is the only method of verifying a theory, it is crucial that we find some observable quantity that inflation predicts and thus either make it a more likely candidate or be done with it altogether.
Power spectra are among the favourite observable quantities of cosmologists.

4.1. Power Spectra

Consider

$$\langle\frac{\delta\rho}{\rho}\rangle^2 \equiv \int P_s^2(k)d(\ln k) \tag{80}$$

Then $P(k)^2$ is the power spectrum and

$$P(k)^2 \propto \delta_k^2, \tag{81}$$

save some normalisation coefficients.
The CMB's power spectrum is often decomposed into spherical harmonics $Y_\ell^m(\theta,\phi)$ rather than into its Fourier components.
Let's introduce the *scalar* power spectrum for the density perturbations:

$$P_s^2(k) = \frac{k^3\langle|\delta_k|^2\rangle}{2\pi^3}; \qquad \delta_k = \frac{1}{\sqrt{(2\pi)^3}}\int d\vec{x}\left(\frac{\delta\rho}{\rho}\right)e^{-i\vec{k}\cdot\vec{x}} \tag{82}$$

again, this is evaluated when the modes leave the Hubble radius; this means

$$P_s^2(k)\big|_{k=aH} = \frac{1}{24\pi^2}\frac{V}{\epsilon}\bigg|_{k=aH} \tag{83}$$

Usually, the power spectrum is taken to be a power law in k [17], i.e. $P_s^2(k)\big|_{k=aH} \propto k^{n_s-1}$, where

$$n_s - 1 \equiv \frac{d\ln P_s^2}{d\ln k} \approx 2\eta - 6\epsilon \tag{84}$$

If $n_s = 1$, we have a scale-invariant power-spectrum, the so-called Harrison-Zel'dovich spectrum; this would *not* agree with inflation, since inflation did not go on forever, meaning the initial perturbations were only produced on certain scales.

Similarly, in addition to the matter density to be perturbed the metric will also receive perturbations; remember the underlying non-linearity of general relativity!
This gives rise to the *tensor* power spectrum:

$$P_T^2(k)\big|_{k=aH} = \frac{2}{3\pi^2}V\bigg|_{k=aH} \tag{85}$$

Again, we assume a power law: $P_T^2(k)\big|_{k=aH} \propto k^{n_T}$
The tensor spectral index thus is, analogous to n_s:

$$n_T \equiv \frac{d\ln P_T^2}{d\ln k} \approx -2\epsilon \tag{86}$$

The relative contributions of tensor perturbations to scalar ones is called r:

$$r \equiv \frac{P_T^2(k)}{P_s^2(k)} = 16\epsilon, \tag{87}$$

or, sometimes with the C_ℓ (ℓth multipole coefficient)[18]:

$$R \equiv \frac{C_{\ell,T}}{C_{\ell,s}} \approx 12\epsilon \tag{88}$$

4.2. Measurements

By measuring those values, we can apply more and more constraints on inflationary models; the zoo of models has condensed down in the last few years due to data getting more precise.
One of the big achievements of WMAP was to measure n_s very precisely; the WMAP 7-year data gives us (error is 1σ) [6]:

$$\boxed{n_s = 0.963 \pm 0.012} \tag{89}$$

n_s is 2σ below unity!
This means 98% confidence level, which is an astronomical number for cosmology. Inflation actually seems to have left traces in our observable universe. But however many signs are pointing towards inflation, we cannot be sure that something else caused these primordial perturbations. But this might change in the near future, since the Planck satellite has only shortly been launched and the data is expected to have unprecedented precision for cosmology. From the CMBR's polarisation, tensor perturbations might be detectable.

5. Conclusions

To summarise this thesis, let me go through each point again.
First, I gave a short introduction to cosmology; stating that only recently has cosmology become a quantitative science. Without general relativity there can be no physical cosmology.
Then, we uncovered some unresolved questions that the ΛCDM model inspires us to ask. These can elegantly be solved by introducing a phase of inflation at very early times, where the universe grew by a factor of $\sim e^{60}$.
This very mechanism can also be used to stretch quantum scales to macroscopic dimensions, mapping quantum fluctuations as density perturbations to the universe. Another question can be answered: Where did the initial density perturbations come from?
Finally, we introduced the power spectra and assumed that they were following a power law. The exponent is a measurable quantity, which makes inflation, in a sense, observable.
We still know very little about the nature of inflation and its cause, yet the Inflaton scalar field is quite a good candidate. However we're lacking a counter-part in particle physics. The only fundamental scalar field there is the Higgs-Field, which has not experimentally been proven yet.
Furthermore, it is still a mystery why the universe has a knack for accelerated expansion; it is doing it again today! It is not at all known where the Cosmological Constant, or Dark Energy, is coming from. It is not the full amount vacuum energy density; the discrepancy is $\mathcal{O}(10^{120})$!
Why is Dark Energy $\mathcal{O}(\rho_{crit})$?
There are still a lot of white patches on cosmology's charts.
For now, however, inflation gives us a highly satisfying answer for the origin of primordial density perturbations. They seem to be the consequence of very fundamental physical processes, showing us again how strangely simple nature really is.

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A. Appendix

A.1. Supplementary Material

The gnuplot source for Fig. 2 can be found at
`http://www.rzuser.uni-heidelberg.de/~ttugendh/inflation/`,
along with the other figures.

A.2. Mathematica Source for Fig. 2

ρ[a_]:=(Ωr0 * ρcr0 * a^ − 4 + Ωm0 * ρcr0 * a^ − 3 + (1/(8 * π * G)) * Ωl0 * 3 * H0^2) + Ωk * H0^2 * a^ − 2
G = 6.67*^-8
h = 0.73
Ωr0 = 4.67*^-5
Ωm0 = 0.26
Ωl0 = 0.76
Ωk = −0.09
ρcr0 = (1.86*^-29) * h^2
H0 = (3.22*^-18) * h
ρ[1]
1.009252681114918*^-29
ρ[a]

$$7.514959026272348\text{*^-30} + \frac{4.628875979999999\text{*^-34}}{a^4} + \frac{2.5771043999999996\text{*^-30}}{a^3} + \frac{4.972788323999998\text{*^-37}}{a^2}$$

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